TOPOLOGICAL CHARACTERIZATIONS FOR NON-WANDERING SURFACE FLOWS

TOMOO YOKOYAMA

Abstract. Let v be a continuous flow with arbitrary singularities on a compact surface M. Then we show that if v is non-wandering, then $\mathrm{Per}(v)$ is open and there are no exceptional orbits. Moreover, v is non-wandering if and only if the closure of the union of locally dense orbits and $\mathrm{Per}(v)$ contains $M - \mathrm{Sing}(v)$.

1. Introduction and preliminaries

In [D], it is shown that there are no exceptional minimal sets of C^2-flows on tori. In [S], this result is generalized to the compact surface cases. On the other hand, there is the Denjoy flow (first constructed by Poincaré [P]) which has an exceptional minimal set. Meanwhile, in [NZ], they have shown the characterization of the non-wandering flows on compact surfaces with finitely many singularities. In [Mar], one has given a description near orbits of the non-wandering flow with the set of singularities which is totally disconnected. In this paper, we show the non-existence of exceptional minimal sets of continuous non-wandering flows with arbitrary singularities on compact surfaces and study the topological characterization of non-wandering flows.

By flows, we mean continuous $\mathbb{R}$-actions on compact surfaces. Let $v : \mathbb{R} \times M \to M$ be a flow on a surface M. Put $v_t(\cdot) := v(t, \cdot)$ and $O_v(\cdot) := v(\mathbb{R}, \cdot)$. Recall that a point x of M is non-wandering (resp. recurrent) if for each neighbourhood U of x and each positive number N, there is $t \in \mathbb{R}$ with $|t| > N$ such that $v_t(U) \cap U$ (resp. $v_t(x) \cap U$) is not empty, and that v is non-wandering if every point is non-wandering. An orbit is proper if it is embedded, locally dense if the closure of it has nonempty interior, and exceptional if it is neither proper nor locally dense. A point is proper (resp. locally dense, exceptional) if so is its orbit. Denote by $\Omega(v)$ the set of non-wandering points of v and by $\mathrm{LD} = \mathrm{LD}(v)$ (resp. E, P) the union of locally dense orbits (resp. exceptional orbits, non-singular non-periodic proper orbits). A recurrent orbit is non-trivial if it is not closed.

2. Topological characterizations of non-wandering surface flows with arbitrary singularities

First, we state the relation between exceptional and proper orbits. Recall that a subset S of a surface M is essential if S is not contractible in M.

Date: December 19, 2012.

The author is partially supported by the JST CREST Program at Department of Mathematics, Hokkaido University.

Lemma 2.1. *Let v a flow on a surface M. Then* $\overline{\mathrm{Sing}(v) \sqcup \mathrm{Per}(v) \sqcup \mathrm{LD}} \cap \mathrm{E} = \emptyset$. *Moreover* $\mathrm{E} \subset \mathrm{int}(\overline{\mathrm{P}})$.

Proof. By taking a double covering of M and the doubling of M if necessary, we may assume that v is transversally orientable and M is closed. For any orbit O with $\overline{O} \cap \mathrm{Per}(v) \neq \emptyset$, the flow box theorem implies $O \subseteq \mathrm{P} \sqcup \mathrm{Per}(v)$. First we show that $\overline{\mathrm{Per}(v)} \cap \mathrm{E} = \emptyset$. Otherwise there is an sequence of periodic orbits O_i of whose closure $A := \overline{\cup_i O_i}$ of the union contains a point $x \in \mathrm{E}$. Then $\overline{O(x)} \cap \mathrm{Per}(v) = \emptyset$. We show that there is $K \in \mathbb{Z}_{>0}$ such that O_k is contractible in M_K for any $k > K$ where M_K is the resulting closed surface of adding $2K$ disks to $M - (O_1 \sqcup \cdots \sqcup O_K)$. Indeed, we may assume that M is not a sphere and O_1 is essential by renumbering. Let M_1 be the resulting surface of adding two center disks to $M - O_1$. Then $g(M_1) < g(M)$, where $g(N)$ is the genus of a surface N. Since M is a compact surface, by induction for essential closed curves at most $g(M)$ times, the assertion is followed. Since M is normal, there are open disjoint neighbourhoods U_x and V of $\overline{O(x)}$ and $\cup_{i \leq K} O_i$. Then there is a transverse arc $\gamma \subset U_x$ through x which does not intersect $\cup_{i \leq K} O_i$. Since x is exceptional, we have x is a recurrent point and so there is an arc γ' in $O_v(x)$ whose boundaries are contained in γ. Since M is normal, there is a neighbourhood of $\gamma \cup \gamma'$ which does not intersect $\cup_{i \leq K} O_i$. Then we can construct a closed transversal T for v through x which does not intersect $\cup_{i \leq K} O_i$. Let v_K be a resulting flow on M_K by adding center disks. Then T is also a closed transversal for v_K. However there is $k > K$ such that T intersects O_k which is contractible in M_K. This is impossible. Second, we show that $\overline{\mathrm{LD}} \cap \mathrm{E} = \emptyset$. Otherwise the Maĭer Theorem [Mai] implies that there is a point $y \in \mathrm{LD}$ with $\overline{O_v(y)} \cap \mathrm{E} \neq \emptyset$. If there is a point $z \in \mathrm{E}$ such that $y \in \overline{O_v(z)}$, then z is locally dense, which contradicts. Thus $\mathrm{LD} \cap \overline{\mathrm{E}} = \emptyset$. Then there is a neighbourhood $U \subseteq M - \overline{\mathrm{E}}$ of y. Fix a transversal C' in U through y. Since y is locally dense, we have that y is recurrent and so there is an arc C in $O_v(y)$ whose ends connect C' such that $C \cup C'$ is a simple closed curve. Since M is normal, there is a neighbourhood V of $C \cup C'$ which does not intersect $\overline{\mathrm{E}}$. By perturbing this closed curve, we can obtain a closed transversal C'' through y which does not intersect $\overline{\mathrm{E}}$. Then $C'' \cap O_v(y)$ is dense in C''. Recall the following fact [BS] that for any countable dense subsets A and B of $\mathbb{R}$, there is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(A) = B$. Thus we can perturb smoothly v near this closed transversal by a rotation such that the resulting time one map makes $C'' \cap O_v(y)$ periodic. Then we obtain a new flow v' such that $\overline{\mathrm{Per}(v')} \supseteq \overline{O_v(y)}$ contains exceptional points for v'. This is a contradiction. The fact $\mathrm{Sing}(v)$ is closed implies the first assertion. Since $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v) \sqcup \mathrm{LD} \sqcup \mathrm{P} \sqcup \mathrm{E}$, we obtain E is contained by a neighbourhood which consists of $\mathrm{P} \sqcup \mathrm{E}$. By the Maĭer Theorem [Mai], E consists of finitely many distinct closures of exceptional orbits and so is nowhere dense. Therefore $\mathrm{E} \subset \mathrm{int}(\overline{P})$. □

From now on, we consider only non-wandering flows.

Lemma 2.2. *Let v a non-wandering flow on a surface M. Then* $\mathrm{Per}(v)$ *is open,* $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v) \sqcup \mathrm{LD} \sqcup \mathrm{P}$, *and* $\overline{\mathrm{LD} \sqcup \mathrm{Per}(v)} \supseteq M - \mathrm{Sing}(v)$.

Proof. By taking a double covering of M if necessary, we may assume that v is transversally orientable. By Theorem III.2.12, III.2.15 [BS2], the set of recurrence points is dense in M. Let $U := M - \overline{\mathrm{Sing}(v) \sqcup \mathrm{Per}(v) \sqcup \mathrm{LD}}$. Then $U \subseteq \mathrm{P} \sqcup \mathrm{E}$ is open. Since each point of P is not recurrent, we have $\overline{\mathrm{E}} \supseteq U$. Since E is nowhere dense, we

have U is empty and so is E. Hence $M = \mathrm{Sing}(v) \sqcup \mathrm{Per}(v) \sqcup \mathrm{LD} \sqcup \mathrm{P}$. Since v is non-wandering, we have that there are no limit cycles and so $\overline{\mathrm{LD} \sqcup \mathrm{P}} \cap \mathrm{Per}(v) = \emptyset$. Since $\mathrm{Sing}(v)$ is closed, we obtain $M - \mathrm{Per}(v) = \mathrm{Sing}(v) \sqcup \mathrm{LD} \sqcup \mathrm{P}$ is closed and so $\mathrm{Per}(v)$ is open. For any $x \in \mathrm{P}$, since the regular recurrent points forms $\mathrm{Per}(v) \sqcup \mathrm{LD}$, by non-wandering property, each neighborhood of x meets $\mathrm{Per}(v) \sqcup \mathrm{LD}$ and so $\overline{\mathrm{LD} \sqcup \mathrm{Per}(v)} \supseteq \mathrm{P}$. $\square$

Now we state the characterization of non-wandering flows.

Theorem 2.3. *Let v be a continuous flow on a compact surface M. Then v is non-wandering if and only if $\overline{\mathrm{LD} \sqcup \mathrm{Per}(v)} \cup \mathrm{Sing}(v) = M$. In particular, if v is non-wandering, then $\mathrm{Per}(v)$ is open and there are no exceptional orbits.*

Proof. Suppose that v is non-wandering. By Lemma 2.2, we have $\overline{\mathrm{LD} \sqcup \mathrm{Per}(v)} \cup \mathrm{Sing}(v) = M$. Conversely, suppose that $\overline{\mathrm{LD} \sqcup \mathrm{Per}(v)} \cup \mathrm{Sing}(v) = M$. For any regular point x of M, we have $x \in \overline{\mathrm{LD} \sqcup \mathrm{Per}(v)}$. This shows that v is non-wandering. $\square$

Recall that a subset is saturated if it is a union of orbits. Consider the case that $\mathrm{Sing}(v)$ is finite.

Corollary 2.4. *Let v be a continuous non-wandering flow on a compact surface M. Suppose that $\mathrm{Sing}(v)$ is finite. Then LD is open. Moreover, for a regular orbit O, there is a saturated neighborhood U of O such that each connected component of $U \setminus \mathrm{P}$ is open and contained in either LD or $\mathrm{Per}(v)$.*

Proof. The non-wandering property implies that $\mathrm{LD} \cap \overline{\mathrm{Per}(v)} = \emptyset$ and that the omega (resp. alpha) limit set of each point of P is a singular point. By the finiteness of $\mathrm{Sing}(v)$, we have $\mathrm{Sing}(v) \sqcup \mathrm{P}$ is closed and so is $\mathrm{Per}(v) \sqcup \mathrm{Sing}(v) \sqcup \mathrm{P} = M - \mathrm{LD}$. Thus LD is open. Applying the flow box theorem to points of P, there is a saturated neighbourhood $U \subseteq M - \mathrm{Sing}(v)$ of O. Since $U \setminus \mathrm{P} \subseteq \mathrm{LD} \cup \mathrm{Per}(v)$, the openness of LD and $\mathrm{Per}(v)$ implies that U is desired. $\square$

References

[BS] Barth, K. F., Schneider, W. J., *Entire functions mapping countable dense subsets of the reals onto each other monotonically* J. London Math. Soc. (2) 2 1970 620–626.

[BS2] Bhatia, N. P., Szeg, G. P., *Stability theory of dynamical systems* Die Grundlehren der mathematischen Wissenschaften, Band 161 Springer-Verlag, New York-Berlin 1970 xi+225 pp.

[D] A. Denjoy, *Sur les courbes définies par les équations differentielles la surface du tore* J. Math. Pures Appl. (9) 11 (1932), 333–375.

[Mai] A. G. Maier, *Trajectories on closed orientable surfaces* Mat. Sb. 12 (54) (1943), 71–84

[Mar] H. Marzougui, *flows with infinite singularities on closed two-manifolds* J. Dyn. Control Syst. 6 (4) (2000) 461–476

[NZ] I. Nikolaev and E. Zhuzhoma, *Flows on 2-Dimensional Manifolds* "Lecture Notes in Mathematics," Vol. 1705, Springer-Verlag, Berlin, 1999.

[P] H. Poincaré, *Sur les courbes définies par une équation différentielle* IV, J. Math. Pures Appl. 85 (1886), 151–217.

[S] Schwartz, A. J., *A generalization of a Poincaré-Bendixson theorem to closed two-dimensional manifolds* Amer. J. Math. 85 (1963), 453–458; errata, ibid 85 1963 753.

Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan,

E-mail address: yokoyama@math.sci.hokudai.ac.jp